Relative Amenability, Amenability, and Coamenability of Coideals

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Locally Compact Quantum Groups

Framework

LCQG $\mathbb{G} = (L^\infty(\mathbb{G}), \Delta_\mathbb{G}, \psi_L, \psi_R)$

- $L^\infty(\mathbb{G}) \subseteq \mathcal{B}(L^2(\mathbb{G}))$ is a vNa;
- $\Delta_\mathbb{G}$ is a vNa coproduct;
- ψ_L and ψ_R are left and right Haar weights respectively.

$L^2(\mathbb{G}) =$ GNS construction from ψ_L.

Locally Compact Quantum Groups

Framework Cont'd

$\exists$ QG C^*-algebras (admit C^*-coproducts):

- $C_0(\mathbb{G})$ - reduced C^*-algebra (wot dense in $L^\infty(\mathbb{G})$);
- $C_0^u(\mathbb{G})$ - universal C^*-algebra.

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Convolution: $f * g = (f \otimes g)\Delta$. Obtain *convolution algebras*:

- $L^1(\mathbb{G}) := L^\infty(\mathbb{G})_*$;
- $C_0^u(\mathbb{G})^*$ - has identity $\epsilon_{\mathbb{G}}^u$ (counit);
- $L^1(\mathbb{G}) \trianglelefteq C_0(\mathbb{G})^* \trianglelefteq C_0^u(\mathbb{G})^*$.

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- $L^1(\mathbb{G}) \trianglelefteq C_0(\mathbb{G})^* \trianglelefteq C_0^u(\mathbb{G})^*$.

$\exists$ injective contractive homomorphism $\lambda^u_{\mathbb{G}} : C_0^u(\mathbb{G})^* \to \mathcal{B}(L^2(\mathbb{G}))$.

$\exists$ LCQG $\widehat{\mathbb{G}}$ such that $L^\infty(\widehat{\mathbb{G}}) = \overline{\lambda_{\mathbb{G}}(L^1(\mathbb{G}))}^{wot}$.

Pontryagin duality: $\hat{\hat{\mathbb{G}}} = \mathbb{G}$.

Classical Examples

Framework Cont'd

Def: CQGs and DQGs:

- $\mathbb{G}$ is compact if $\psi_L(1) < \infty \implies \psi_L = \psi_R = h_{\mathbb{G}} \in L^1(\mathbb{G})$.
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Locally Compact Groups (LCGs)

G - LCG:

- $G = (L^\infty(G, m_L), \Delta_G, m_L, m_R)$, $\Delta_G(x)(s,t) = x(st)$ m_L-a.e..
- $\widehat{G} = (VN(G), \Delta_{\widehat{G}}, \psi)$. When G is discrete, $\psi = 1_{\{e\}}$, $1_{\{e\}}(\lambda(s)) = \delta_{s,e}$.

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- $C(\widehat{G}) = C_r^*(G)$, $C^u(\widehat{G}) = C^*(G)$.

Coideals

Definition

A **coideal** of $\mathbb{G}$ is a $\mathbb{G}$-invariant vN subalgebra $N \subseteq L^\infty(\mathbb{G})$:

$$\Delta_{\mathbb{G}}(N) \subseteq N \overline{\otimes} L^\infty(\mathbb{G}).$$

Write $N \leq L^\infty(\mathbb{G})$.

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Coduals (Izumi-Longo-Popa '98)

Let $N \leq L^\infty(\mathbb{G})$. Then

$$\widetilde{N} := N' \cap L^\infty(\widehat{\mathbb{G}})$$

is a coideal called the **codual** of N. We have $\widetilde{\widetilde{N}} = N$.

Coideals

Definition

A **group-like projection** is $P \in L^\infty(\mathbb{G})$ such that $P^* = P^2 = P$ and

$$(P \otimes 1)\Delta_{\mathbb{G}}(P) = P \otimes P.$$

Let $GProj(L^\infty(\mathbb{G}))$ = group-like projections. Also,

$$\widetilde{N_P} = \{x \in L^\infty(\mathbb{G}) : (P \otimes 1)\Delta_{\mathbb{G}}(x) = P \otimes x\} \leq L^\infty(\mathbb{G}).$$

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Theorem (Kasprzak '18, Kasprzak-Khosravi-Sołtan '18)

Let $\mathbb{G}$ be a CQG and $N \leq L^\infty(\mathbb{G})$. Let $PL^2(\mathbb{G}) = L^2(N)$ where $P^2 = P^* = P \in \mathcal{B}(L^2(\mathbb{G}))$.

- $P \in \widetilde{N} \cap GProj(\ell^\infty(\widehat{\mathbb{G}}))$;
- $\widetilde{N} = \widetilde{N_P}$.

Coideals

"Compact" Coideals - (...,Salmi-Skalski '09,..., Ilie-Spronk '05, Host '86, Kawada-Itô '80, Cohen '60)

- Let $Idem(C_0^u(\mathbb{G})) \subseteq C_0^u(\mathbb{G})^*$ denote the idempotent states.

  If $\omega \in C_0^u(\mathbb{G})^*$ then $P_\omega := \lambda_{\mathbb{G}}^u(\omega) \in GProj(L^\infty(\widehat{\mathbb{G}}))$ (Faal-Kasprzak '17).
- $N_\omega = N_{P_\omega}$ is called a **compact quasi-subgroup**.

Quantum Subgroups

Definition: Quantum Subgroups (Vaes)

We say $\mathbb{H} \leq \mathbb{G}$ if $L^\infty(\widehat{\mathbb{H}}) \subseteq L^\infty(\widehat{\mathbb{G}})$ and $\Delta_{\widehat{\mathbb{H}}} = \Delta_{\widehat{\mathbb{G}}}|_{L^\infty(\widehat{\mathbb{H}})}$.

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Let $\mathbb{G}$ be a CQG:

$L^\infty(\mathbb{H}\backslash\mathbb{G}) = N_{\omega_{\mathbb{H}}}$ for some $\overbrace{\omega_{\mathbb{H}}}^{\text{Haar idempotent}} \in Idem(C^u(\mathbb{G}))$. Moreover,

$$N_\omega = L^\infty(\mathbb{H}\backslash\mathbb{G}) \iff \{a \in C^u(\mathbb{G}) : \omega(a^*a) = 0\} \trianglelefteq C^u(\mathbb{G})$$

(Salmi-Skalski '16).

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(Salmi-Skalski '16). Also,

$$\ell^\infty(\widehat{\mathbb{H}}\backslash\widehat{\mathbb{G}}) = \widetilde{N_P} \iff P = P_{\widehat{\mathbb{H}}} \in Z(\ell^\infty(\widehat{\mathbb{G}}))$$

(Kalantar-Kasprzak-Skalski '16).

Examples

Classical Case

Let K be a compact group.

- $P \in GProj(VN(K)) \iff P = \int_H \lambda_G(s)\, dm_H(s)$, $H \leq K$.
- $\omega \in Idem(C(K)) \iff \omega = m_H$, $H \leq K$ (Kl '80).
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- $P \in GProj(\ell^\infty(G)) \iff P = 1_H$, $H \leq G$.
- $\omega \in Idem(C^*(G)) \iff \lambda^u_{\widehat{G}}(\omega) = 1_H$, $H \leq G$ (IS'05).
- $\mathbb{H} \leq \widehat{G} \iff \mathbb{H} = \widehat{N\backslash G}$, $N \trianglelefteq G$ (not every coideal is a quotient).

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- $\mathbb{H} \leq \widehat{G} \iff \mathbb{H} = \widehat{N\backslash G},\ N \trianglelefteq G$ (not every coideal is a quotient).

Note: not every coideal is "compact". Eg. non-standard Podleś spheres of $SU_q(2)$.

Amenability

Definition

$\mathbb{G}$ is **amenable** if $\exists$ state $L^\infty(\mathbb{G}) \to \mathbb{C}$ s.t. $m * f = m(\mathrm{id} \otimes f)\Delta_{\mathbb{G}} = f(1)m$, $f \in L^1(\mathbb{G})$.

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A ucp map $\Psi : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ is $\mathbb{G}$-**equivariant** if $(\Psi \otimes \mathrm{id})\Delta = \Delta \circ \Psi$.

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Definition

$N \leq L^\infty(\mathbb{G})$ is **relatively amenable** if there exists a $\mathbb{G}$-equivariant ucp map $\Psi : L^\infty(\mathbb{G}) \to N$.

$N \leq L^\infty(\mathbb{G})$ is **amenable** if there exists a $\mathbb{G}$-equivariant ucp map $L^\infty(\mathbb{G}) \to N$, $\Psi|_N = \mathrm{id}_N$.

Amenability

Given $\mu \in C_0^u(\mathbb{G})^*$, let $L_\mu : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ denote the associated left multiplier (normal $\mathbb{G}$-equivariant map).

Remark

- For $\omega \in Idem(C_0^u(\mathbb{G}))$, $L_\omega : L^\infty(\mathbb{G}) \to N_\omega$ is a *normal* $\mathbb{G}$-equivariant ucp map such that $L_\omega|_{N_\omega} = \mathrm{id}_{N_\omega}$.
- When $\mathbb{G}$ is a CQG and $\omega \in L^1(\mathbb{G}) \cap Idem(C_0^u(\mathbb{G}))$, then N_ω is finite dimensional (Sołtan-Kasprzak '20).

Amenability

Note: when $\widehat{\mathbb{H}} \leq \widehat{\mathbb{G}}$, $\ell^1(\widehat{\mathbb{H}}) \subseteq \ell^1(\widehat{\mathbb{G}})$. Then, $\widehat{\mathbb{H}}$-invariant means $m * \varphi = \varphi(1)m$ for all $\varphi \in \ell^1(\widehat{\mathbb{H}})$.

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Classical Case (Caprace-Monod '14)

$H \leq G$. TFAE

1. H is amenable;
2. $\ell^\infty(H \backslash G)$ is amenable;
3. $\ell^\infty(H \backslash G)$ is relatively amenable;
4. $\ell^\infty(G)$ has an H-invariant state.

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Theorem (Kalantar-Kasprzak-Skalski-Vergnioux '20)

TFAE:

1. $\widehat{\mathbb{H}}$ is amenable;
2. $\ell^\infty(\widehat{\mathbb{H}}\backslash\widehat{\mathbb{G}})$ is relatively amenable;
3. $\exists$ $\widehat{\mathbb{H}}$-invariant state $\ell^\infty(\widehat{\mathbb{G}}) \to \mathbb{C}$.

Amenability

Open Problem

Let G be a locally compact group. Does relative amenability of $L^\infty(H\backslash G)$ imply amenability of $L^\infty(H\backslash G)$?

Amenability

Denote the weak* closed $\widehat{\mathbb{G}}$-invariant operator system

$$M_P = \{x \in \ell^\infty(\widehat{\mathbb{G}}) : (P \otimes 1)\Delta_{\widehat{\mathbb{G}}}(x)(P \otimes 1) = P \otimes x\} \supseteq \widetilde{N_P}.$$

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Given $x \in \ell^\infty(\mathbb{G})$, denote xf, fx s.t. $(xf)(y) = f(yx)$ and $(fx)(y) = f(xy)$.

Theorem (A-S)

1. $\widetilde{N_P}$ is relatively amenable iff there exists a state $m : \ell^\infty(\widehat{\mathbb{G}}) \to \mathbb{C}$ s.t. $m * (fP) = f(P)m$ for all $f \in \ell^1(\widehat{\mathbb{G}})$.

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2. M_P is amenable iff there exists a state $m : \ell^\infty(\widehat{\mathbb{G}}) \to \mathbb{C}$ s.t. $m * (PfP) = f(P)m$ and $m(P) \neq 0$ for all $f \in \ell^1(\widehat{\mathbb{G}})$.

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3. If there exists a state $m : \ell^\infty(\widehat{\mathbb{G}}) \to \mathbb{C}$ s.t. $m * (fP) = f(P)m$ and $m(P) \neq 0$ for all $f \in \ell^1(\widehat{\mathbb{G}})$ then M_P is amenable .

Amenability

Question: starting with a unital cb $\mathbb{G}$-equivariant map $\Psi : L^{\infty}(\mathbb{G}) \to N$, can we obtain one that is ucp?

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Let A be a Banach algebra that has a bounded approximate identity (eg. is unital). Let $I \subseteq A$ be a closed right ideal. Denote

$$I^\perp = \{\varphi \in A^* : \varphi|_I = 0\}.$$

Theorem (Forrest '87)

I has a bounded left approximate identity (blai) if and only if there is a left A-module map $\Psi : A \to I^\perp$ such that $\Psi|_{I^\perp} = \mathrm{id}_{I^\perp}$.

Amenability

Note: $m \in \ell^\infty(\widehat{\mathbb{G}})^*$ is $\ell^1(\widehat{\mathbb{G}})P$-invariant iff $\epsilon_{\widehat{\mathbb{G}}} - m \in \ell^\infty(\widehat{\mathbb{G}})^*$ is a left identity for the closed right ideal $(\widetilde{N_P})_\perp \subseteq \ell^1(\widehat{\mathbb{G}})$. Then we can obtain a blai for $(\widetilde{N_P})_\perp$ in $(\mathbb{C}1)_\perp \supseteq (\widetilde{N_P})_\perp$.

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Working a little harder, we obtain...

Theorem (Caprace-Monod '14)

Let G be a discrete group. $H \leq G$ is amenable iff $\ell^\infty(H\backslash G)_\perp$ has a blai.

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Let G be a discrete group. $H \leq G$ is amenable iff $\ell^\infty(H\backslash G)_\perp$ has a blai.

Theorem (A-S)

Let $\mathbb{G}$ be a CQG. $M_P \subseteq \ell^\infty(\widehat{\mathbb{G}})$ is amenable iff $(M_P)_\perp$ has a blai.

Amenability

Definition $\mathbb{G}$-LCQG

$\mathbb{G}$ is **coamenable** if $C_0^u(\mathbb{G}) = C_0(\mathbb{G}) \iff \epsilon_{\mathbb{G}} \in C_0(\mathbb{G})^*$.

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Tomatsu '06: A CQG $\mathbb{G}$ is coamenable $\iff$ $\widehat{\mathbb{G}}$ is amenable.

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Classical Case

Let $H \leq G$. Then

$$H \text{ is amenable} \iff \widehat{H} \text{ is coamenable (Hulanicki '64-'66)}$$
$$\iff 1_H \in C(\widehat{G})^*.$$

Amenability

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Given $P \subseteq GProj(\ell^\infty(\widehat{\mathbb{G}}))$, let

$$C(N_P) = \overline{\lambda_{\widehat{\mathbb{G}}}(\ell^1(\widehat{\mathbb{G}})P)} \subseteq C(\mathbb{G}) \text{ and } C^u(N_P) = \overline{\lambda_{\widehat{\mathbb{G}}}(\ell^1(\widehat{\mathbb{G}})P)} \subseteq C^u(\mathbb{G}).$$

Note $\overline{C(N_P)}^{wot} = N_P$ (Kasprzak '18) and $C(N_P)^* \subseteq C^u(N_P)^*$.

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Note $\overline{C(N_P)}^{wot} = N_P$ (Kasprzak '18) and $C(N_P)^* \subseteq C^u(N_P)^*$.

Proposition / Definition of Coamenability

The following are equivalent:

- $\exists$ a state $\epsilon_P \in C(N_P)^*$ such that $\epsilon_P = \epsilon^u_{\mathbb{G}}|_{C^u(N_P)}$;
- $C(N_P) = C^u(N_P)$.

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Remark

- If $\widehat{\mathbb{H}} \leq \widehat{\mathbb{G}}$ then $\mathbb{H}$ is coamenable iff $L^\infty(\mathbb{H})$ is coamenable.
- $L^\infty(\mathbb{H}\backslash\mathbb{G})$ is coamenable iff $\omega_{\mathbb{H}} \in C(\mathbb{G})^*$ can be easily obtained from work of KKSV'20.

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Corollary (A-S)

Let $\widehat{\mathbb{H}} \leq \widehat{\mathbb{G}}$. TFAE:

1. $\widehat{\mathbb{H}}$ is amenable;
2. $\ell^\infty(\widehat{\mathbb{H}}\backslash\widehat{\mathbb{G}})$ is amenable;
3. $\ell^\infty(\widehat{\mathbb{H}}\backslash\widehat{\mathbb{G}})_\perp$ has a blai.

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Open Problems

Determine if TFAE:

1. Relative amenability of $\widetilde{N_P}$;
2. Amenability of $\widetilde{N_P}$;
3. Amenability of M_P;
4. Coamenability of N_P.

Amenability

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Positive Answer Summary

- 2. $\implies$ 1. (by definition).
- 4. $\implies$ 3. when $P = P_\omega$.
- 1. $\iff$ 2. $\iff$ 3. $\iff$ 4 when $P = P_{\widehat{\mathbb{H}}} \in Z(\ell^\infty(\widehat{\mathbb{G}}))$.
- 4. $\implies$ 1. when $P = P_{\omega_{\mathbb{H}}}$ and $\omega_{\mathbb{H}}$ is tracial (to be discussed next).

Tracial States

Given a C^*-algebra $A \subseteq \mathcal{B}(\mathcal{H})$, a **tracial state** is a state $\tau : A \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$.

Notation: $T(A) =$ tracial states on A.

Operator Algebraic Aspects

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Examples

- $A = C_r^*(G)$ and $\tau = 1_N$ where $N \trianglelefteq G$ is amenable.

  Eg. $1_{\{e\}} = h_{\widehat{G}}$, $1_{R_a(G)}$, where $R_a(G)$ is the amenable radical.

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- $A = C(\widehat{\mathbb{G}})$, $h_{\widehat{\mathbb{G}}}$ not always tracial. In fact, $T(C(\mathbb{G})) = \varnothing$ is possible (eg. the non-Kac U_F^+, O_F^+).

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Definition

$\mathbb{G}$ of **Kac type** if the Haar state $h_{\mathbb{G}}$ is tracial.

Operator Algebraic Aspects

Definition

A C^*-algebra is **nuclear** if $A \otimes_{min} B = A \otimes_{max} B$ for every C^*-algebra B.

Theorems

Let $\mathbb{G}$ be a LCQG.

- If G is a discrete group, then G is amenable iff $C_r^*(G)$ is nuclear (Lance '73).

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- $\mathbb{G}$ is coamenable iff $C(\mathbb{G})$ is nuclear and admits a $\widehat{\mathbb{G}}$-invariant state (Crann '17, Ng-Viselter '17, Bédos-Tuset '03).
- When $\mathbb{G}$ is Kac type and compact, a state is tracial iff it is $\widehat{\mathbb{G}}$-invariant (NV'17, KKSV'20).
- $\implies$ when $\mathbb{G}$ is Kac type and compact, $\mathbb{G}$ is coamenable iff $C(\mathbb{G})$ is nuclear.

Operator Algebraic Aspects

Tracial Idempotents

- Let $\omega \in Idem(C^u(\mathbb{G}))$

$$\omega \in T(C(\mathbb{G})) \iff \omega = \omega_{\mathbb{H}},\ \mathbb{H} \text{ is Kac \& } \mathbb{H}\backslash\mathbb{G} \text{ is coamenable.}$$

- Using convolution powers and compactness:

$$T(C(\mathbb{G})) \neq \varnothing \iff Idem(C(\mathbb{G})) \cap T(C(\mathbb{G})) \neq \varnothing.$$

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Theorem (A-S)

Let $\mathbb{G}$ be a CQG. $\mathbb{G}$ is coamenable iff $C(\mathbb{G})$ is nuclear and has a tracial state.

Operator Algebraic Aspects

Proof

Idea: reduce to Kac type case.

$\mathbb{G}$ is coamenable $\implies C(\mathbb{G})$ is nuclear is due to BT'03 and $\epsilon_{\mathbb{G}} \in C(\mathbb{G})^*$ is tracial.

Operator Algebraic Aspects

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Assume $C(\mathbb{G})$ is nuclear and admits a tracial state.

$T(C(\mathbb{G})) \neq \varnothing \implies \exists\, \omega_{\mathbb{H}} \in Idem(C(\mathbb{G})) \cap T(C(\mathbb{G}))$, $\mathbb{H} \leq \mathbb{G}$ ($\mathbb{H}$ is Kac type and $\mathbb{H}\backslash\mathbb{G}$ is coamenable).

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$\mathbb{H}\backslash\mathbb{G}$ is coamenable $\implies$ $C(\mathbb{H}) = C(\mathbb{G})/I$ (KKSV'20) $\implies$ $C(\mathbb{H})$ is nuclear $\implies$ $\mathbb{H}$ is coamenable (NV'17).

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$\mathbb{H}$ is coamenable and $\mathbb{H}\backslash\mathbb{G}$ is coamenable $\implies$ $\mathbb{G}$ is coamenable (KKSV'20).

Operator Algebraic Aspects

Classical Results

- $C_r^*(G)$ is simple $\iff$ no amenable residually normal subgroups. (Kennedy '20)
- $C_r^*(G)$ has unique trace $\iff$ no amenable normal subgroups ($R_a(G) = \{e\}$). (Breuillard-Kalantar-Kennedy-Ozawa '14, Kalantar-Kennedy '14)

Operator Algebraic Aspects

Note: $\ell^\infty(R_a(G)\backslash G) \subseteq \ell^\infty(N\backslash G)$ whenever $N \trianglelefteq G$ and is amenable.

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Normal Quantum Subgroups

Let $\mathbb{G}$ be a LCQG. $\mathbb{H} \leq \mathbb{G}$ is **normal** when

$$\Delta_{\mathbb{G}}(L^\infty(\mathbb{H}\backslash\mathbb{G})) \subseteq L^\infty(\mathbb{H}\backslash\mathbb{G})\overline{\otimes}L^\infty(\mathbb{H}\backslash\mathbb{G}).$$

$\mathbb{H}$ is normal iff $\widehat{\mathbb{H}\backslash\mathbb{G}} \trianglelefteq \widehat{\mathbb{G}}$.

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Furstenberg Coideal

- $\exists\ \mathbb{G}_F \leq \mathbb{G}$ such that $\mathbb{G}_F \leq \mathbb{H} \leq \mathbb{G}$ whenever $\ell^\infty(\widehat{\mathbb{H}}) \subseteq \ell^\infty(\widehat{\mathbb{G}})$ is relatively amenable. (KKSV'20)

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- $\exists$ a unique largest normal amenable quantum subgroup $R_a(\widehat{\mathbb{G}}) \trianglelefteq \widehat{\mathbb{G}}$ $\implies \ell^\infty(\widehat{\mathbb{G}_F}) \subseteq \ell^\infty(R_a(\widehat{\mathbb{G}})\backslash\widehat{\mathbb{G}})$. (KKSV'20)

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- $\mathbb{G}_F \leq \mathbb{H} \leq \mathbb{G}$ whenever $\mathbb{H}\backslash\mathbb{G}$ is coamenable and $\mathbb{H}$ is Kac type. (A-S)

Operator Algebraic Aspects

Corollary (A-S)

Let $\mathbb{G}$ be a CQG and $\mathbb{H} \leq \mathbb{G}$ be Kac type. If $\mathbb{H}\backslash\mathbb{G}$ is coamenable then $\ell^\infty(\widehat{\mathbb{H}})$ is relatively amenable.

Proof

If $\mathbb{H}\backslash\mathbb{G}$ is coamenable then $\ell^\infty(\widehat{\mathbb{G}_F}) \subseteq \ell^\infty(\widehat{\mathbb{H}})$.

Operator Algebraic Aspects

For a CQG $\mathbb{G}$, we let $\mathbb{G}_{Kac} \leq \mathbb{G}$ denote the largest quantum subgroup of Kac type (Sołtan '05 (attributed to Vaes)).

Advances

- $\mathbb{G}$ Kac type: $\mathbb{G}_F = \mathbb{G} \implies C(\mathbb{G})$ has a unique trace. (KKSV'20)

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- *$\mathbb{G}_F\backslash\mathbb{G}$ is coamenable: $C(\mathbb{G})$ has a unique idempotent tracial state $\iff \mathbb{G}_F = \mathbb{G}_{Kac}$. (A-S)
- *$\mathbb{G}_F\backslash\mathbb{G}$ and $(\mathbb{G}_{Kac})_F\backslash\mathbb{G}_{Kac}$ are coamenable: $C(\mathbb{G})$ has a unique tracial state $\iff \mathbb{G}_F = \mathbb{G}_{Kac}$ and $C^\sigma(\mathbb{G}_{Kac}) = C(\mathbb{G}_{Kac})$. (A-S)

Operator Algebraic Aspects

Open Problem

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Is $\mathbb{G}_F \backslash \mathbb{G}$ coamenable?

Assume $\mathbb{G}_F \backslash \mathbb{G}$ is coamenable. Let $\mathbb{H} \leq \mathbb{G}$ be Kac type. Then

$$\begin{aligned} \mathbb{G}_F \leq \mathbb{H} &\iff \ell^\infty(\widehat{\mathbb{G}_F}) \subseteq \ell^\infty(\widehat{\mathbb{H}}) \\ &\iff \ell^\infty(\widehat{\mathbb{H}}) \text{ is relatively amenable} \end{aligned}$$

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If $\mathbb{G}$ is Kac type and $\mathbb{G}_F \backslash \mathbb{G}$ is coamenable then $C(\mathbb{G})$ has a unique trace $\implies \mathbb{G}_F = \mathbb{G}$.

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